The integration of interior-point methods, decomposition concepts and branch-and-bound to solve large scale MIPs

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Motivation:

Mixed integer programming (MIP) is a powerful modelling tool for decision-making in the industry and in the public sector. Integer requirements are essential to model a wide variety of situations involving assignment restrictions, logical constraints and yes/no decisions, to name a few. Usually real-life applications result in mixed integer programs that are large in size and that are beyond the solution capabilities of the available software. To meet the challenge of solving large scale mixed integer programming problems in reasonable time, there is an urgent need to develop new solution approaches and algorithmic ideas. Large-scale MIP is characterized not only by large size but also by special structure. Structure results from model characteristics such as multi-item, multi-period or multi-echelon. It is through careful exploitation of this feature that efficient solution methodologies are designed.

This article is based on the paper by Elhedhli and Goffin [4] that presents a novel solution approach for large-scale mixed integer programming. The methodology integrates three bodies of research: interior-point methods, decomposition techniques and branch-and-bound approaches (see Figure 1). The integration of classical decomposition concepts and branch-and-bound lead to branch-and-price, an approach that proved very successful in solving large mixed integer programming problems. The approach was initiated by the pioneering work of Gilmore and Gomory on the cutting stock problem [6] and prospered in the context of routing and scheduling by the Desrosiers-Soumis team [3]. Recently, there has been considerable interest in this solution technique. Barnhart et. al. [1] give an overview of the approach describing the different models and branching rules. Vanderbeck and Wolsey [10] develop a new branching rule that generalizes existing ones and that is easily handled within the branch-and-price framework.

The merge of classical decomposition concepts and interior-point methods leads to the analytic center cutting plane method (ACCPM) by Goffin and Vial [5]. ACCPM is a cutting plane method where a subset of cuts are used and the rest are generated when needed. Interior-point techniques are used to calculate a central point at which cutting planes are generated. The method was successfully used to solve a wide variety of large scale problems.

Traditionally, most of the advances in integer programming have closely followed that of linear programming (LP) as linear programs are repeatedly solved within LP-based branch-and-bound methods. Following the initiation of the interior-point field, some attempts were made to substitute interior-point methods for simplex methods, referred to as interior-point branch-and-bound (IP B&B). This substitution was not successful because it is cumbersome to reoptimize an LP using interior-point methods after adding cuts [1].

Motivated by the success that ACCPM and branch-and-price have achieved in solving nondifferentiable optimization and large-scale integer programming problems respectively, and the quest for a method that efficiently integrates interior-point methods and branch-and-bound, we propose to integrate the three techniques into an interior-point branch-and-price (IP B&P) method.

The IP-B&P Method to Solve Large Scale MIPs

The IP-B&P approach works as follows. First, a problem's structure is exploited in a decomposition method. Second, the resulting master problem is solved using an interior-point cutting plane method. Finally, these approaches are incorporated into a branch-and-bound search scheme. To clarify the idea, let us consider the example of scheduling daily operations in a hospital. Since the problem is dynamic, we **decompose** it by day, and schedule each daily operations separately. The coordination of the different daily schedules is done using the **interior-point cutting plane method.** Finally, **branch-and-bound** is used to generate overall feasible (optimal) schedules.

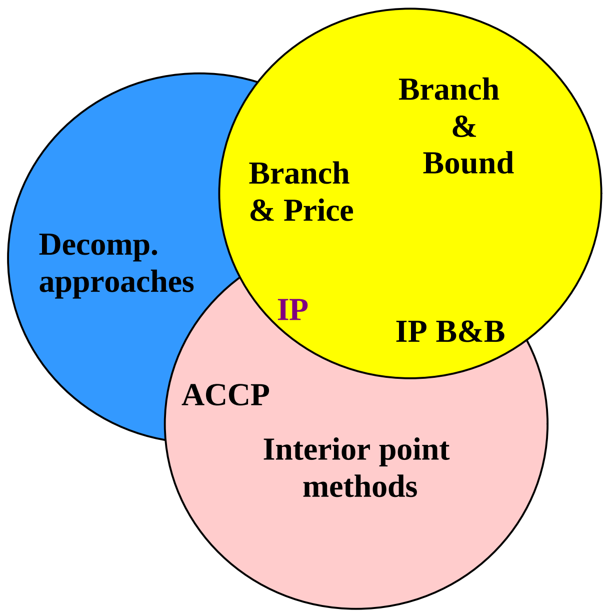

Figure 1: The integration of decomposition concepts, interior-point methods and branch-and-bound.

Branch-and-price is commonly defined as a technique where column generation is used within a branch-and-bound framework. By duality, it is analogous to a Lagrangean-based branch-and-bound where the Lagrangean dual problem is solved using cutting planes. Formulating the Lagrangean dual problem as a linear program yields the dual of the full master problem that is solved at each node of the branch-and-price algorithm. Column generation solves the primal full master problem starting with a restricted problem and adding columns as needed, while cutting plane methods solve the full dual master problem starting with a relaxed problem and appending constraints as necessary.

In a Lagrangean-based branch-and-bound, the predominant task is the solution of the Lagrangean dual problem, which is nondifferentiable. Most of the literature use subgradient optimization. Although simple to implement, subgradient methods are slow to converge and have no clear stopping criteria [1]. Alternatively, the Lagrangean dual problem can be solved using a cutting plane method that is applied to the dual master problem. The cuts are added based on a query point from the relaxed master problem. The choice of the query point distinguishes different variants of cutting plane methods, equivalently, different variants of column generation schemes. Classical branch-and-price methods use a dual extreme point of the restricted master problem as a query point. By duality, this corresponds to Kelley's cutting plane method [8] where cuts are generated at an extreme point of the relaxed master problem. It is known that Kelley's method suffers from tailing effects and that generating cuts at a center of the relaxed master problem's feasible region is superior. The main difficulty with central point strategies resides in the calculation of centers of convex sets. Calculating the center of gravity, for example, is more difficult than optimizing the original problem. The Analytic Center Cutting Plane Method (ACCPM) [5] is designed to overcome this difficulty by generating cuts based on the "analytic center" concept from the interior-point literature. More precisely, the dual full master problem is solved using ACCPM where the cuts are generated at the analytic center of a bounded subset of the dual feasible region.

The IP-B&P approach is essentially a branch-and-bound method with a Lagrangean bounding scheme that is computed using an interior-point cutting plane method. The resulting method is more than the combination of these three different techniques. It addresses and fixes complications that arise as a consequence of this integration. This includes the restarting of the interior-point methods, the branching rule and the exploitation of past information as a warm start. The paper [4] presents the IP-B&P method and details its different components. It discusses the Lagrangean-based lower bound, its efficient computation using ACCPM and the use of dual information from ACCPM to generate incumbent feasible solutions and guide the branching rule. In addition, information, in the form of cuts, incumbent and lower bounds, at the parent node is used to initialize the method at child nodes and to naturally "warm start" the interior-point cutting plane method. As the dual simplex method uses the final tableau at the parent node as an initial basis for the child node, the dual interior-point method uses the final analytic center at the parent node to solve the child node. Computational experience clearly indicates the effectiveness of this strategy.

Finally, it is important to note that there is an abundance of branch-and-bound algorithms that use a Lagrangean bounding scheme and solve the Lagrangean dual using sub-gradient optimization or dual-ascent. Only the approaches that solve the Lagrangean dual using a cutting plane/column generation method do qualify as branch-and-price methods.

From another perspective, this study is a major step in the efficient use of interior-point methods within branch-and-bound approaches for integer programming. Previous attempts [7] have focused on the solution of the linear programs using an interior-point method. Our approach is fundamentally different in two ways. On the one hand, we use a Lagrangean bound rather than the linear-programming bound. On the other hand, the interior-point method is used in a cutting plane context rather than as a direct solution method.

Conclusions and Future Research

This study was motivated by a set of encouraging factors. First, the Lagrangean bound is at least as good as the LP bound and tends to be sharper if a suitable relaxation is used. Second, ACCPM is able to provide proven optimal solution to the Lagrangean dual problem in a reasonable convergence pattern. It does not search blindly as in subgradient optimization and does not show tailing effects as in Kelley's cutting plane method. Third, information in the form of generated cuts, incumbent and lower bounds is efficiently exploited in subsequent nodes both by the search scheme and by ACCPM. Recentering when adding or deleting cuts is done fairly quickly using primal and dual interior-point methods, respectively.

The use of interior-point methods within LP-based branch-and-bound was not successful because of the incompatibility between the LP bounding scheme and the interior-point solution methods.
This study suggests that interior-point methods are naturally suited for branch-and-bound when a Lagrangean bounding scheme is used. The resulting Lagrangean duals are optimally solved using an interior-point cutting plane method.

The IP-B&P methodology extends the current body of research in the fields depicted in Figure 1. It also opens new research venues that will have a positive impact in solving larger instances of real-life problems that were previously out of reach.

For future research, an efficient implementation of the IP-B&P algorithm that uses recent advances in numerical linear algebra and extensive testing on classical MIPs from the literature are the most immediate items on the agenda. Within the algorithm, the use of variable fixing strategies, valid cuts and new branching rules may have an impact on its performance.

Application-wise, we plan to use the interior-point branch-and-price methodology as a basis for solving some classical and more recent large-scale mixed integer programs. Classical problems that are amenable for solution by this approach consist of the many scheduling, routing and transportation problems in the literature. Recent problems consist of the analytical models that arise from the integration of inventory, production and distribution decisions in logistics and supply-chain management.

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