A Brief Outline of the Level Crossing Method in Stochastic Models

Percy H. Brill
Dept. of Math. & Stat. and Dept. of Management Sci.
University of Windsor
Windsor, Ontario
N9B 3P4

1. Introduction

The purpose of this article is to provide a very brief overview of the level crossing method in stochastic models. In addition, it may serve as a brief tutorial on how to apply the method in various stochastic models, such as queues, inventories, dams, risk reserve models in insurance, counter models, etc.

Level crossing methods for obtaining probability distributions in stochastic models, were originated by the present author in 1974 while working on his PhD thesis. In the original thesis, one of the main problems was to derive the steady state *pdf* (probability density function) and *cdf* (cumulative distribution function) of the waiting time in $M/M/c$ queues with service time depending on waiting time, in terms of the input parameters. The method of solution used at that time, started with a Lindley recursion for the customer waiting time. For exposition, consider the simpler Lindley recursion for the $G/G/1$ queue, namely $W_{n+1} = \max\{W_n + S_n - T_{n+1}, 0\}$ $(n \geq 1)$, where W_n and S_n are respectively the waiting time and service time of the nth customer, and T_{n+1} is the interarrival time between the nth and $(n+1)$st customers' arrival epochs. In that approach, the Lindley recursion is utilized as the starting point of a sequence of analytic steps, integrations, differentiations, and algebraic steps, ending in the derivation of an integral equation, or system of integral equations, for the steady state *pdf* of the waiting time. The resulting integral equation is then solved simultaneously with a normalizing condition or other conditions, to obtain the desired *pdf*. In the present article, that approach will be called the *classical* method, which had been applied extensively by various authors since 1952. It turns out, based on considerable personal experience using the classical method, that it may require extensive, tedious, time consuming analysis, in order to move from the Lindley recursion to the desired integral equation for the *pdf*, especially in complex stochastic models with state dependencies.

After applying the classical method repeatedly for a variety of queueing models of varying complexity over roughly a two year period, the question naturally arose as to whether there may exist a *faster* and *easier* method to derive the desired integral equation for the *pdf*, which may bypass the procedure starting from a Lindley recursion. Upon pondering this question while continuing to apply the classical method for an additional year, continually examining the derived integral equations, making conjectures and testing their veracity, the level crossing method gradually evolved and ultimately came to fruition in August, 1974. The details of the stream of ideas underlying the inductive search carried out by the author for the answer to the foregoing question, which ultimately resulted in the level crossing method, will appear elsewhere.

The level crossing method is in fact only one essential component of the more general *system point method.* It is also known as system point theory, system point analysis, sample path analysis, level crossing technique, level crossing approach, level crossing theory, level crossing analysis, etc. in the literature. This overview will present a fairly general stochastic model which commonly occurs in operations research, and will illustrate the application of the level crossing method in a particular example.

2. Model and Stationary Distribution

Consider a stochastic process $\{W(t), t \geq 0\}$ where the state space is continuous, and t represents *time* measured from 0. The random variable $W(t)$ at time point t may, for example, denote the content of a dam with general efflux, the stock on hand in an $< s, S >$ or $< r, nQ >$ inventory system with continuous stock decay, the virtual wait or workload in a queue with complex state dependencies, etc. Assume that upward jumps of $\{W(t)\}$ occur at a Poisson rate λ_u and downward jumps occur at a Poisson rate λ_d. These jumps are assumed to be independent of each other and of the state of the system. Let the corresponding upward and downward jump magnitudes have *cdf*'s B_u and B_d, and define the corresponding complementary *cdf*'s by $\bar{B}_u$ and $\bar{B}_d$, respectively. In some models, other jumps may also be allowable depending on the system state, in accordance with the specific model dynamics. Particular models may admit only one type of jump; other models may allow any two jump types, or all three jump types. Assume that the model parameters are such that the steady state distribution of $W(t)$ exists as $t \to \infty$, and let G and g denote the steady state *cdf* and *pdf* respectively. Our aim is to obtain an integral equation for g, and then to solve this equation for g in terms of the model input parameters. It is then routine to find the expression for G.

An essential idea underlying the level crossing approach, is that the analyst first constructs a typical sample path of the underlying stochastic process. That is, the starting point is from knowledge of a typical sample path of the process. Intuitively, a sample path may be thought of as a typical tracing, or evolution, of the state random variable over time. In many applications, construction of sample paths is straightforward and can be accomplished in a reasonable time – a few minutes to several hours. In complex models with state dependencies, construction of sample paths may be a nontrivial or challenging task. It is important to note that the correct construction of a sample path goes hand in hand with a thorough understanding of the dynamics of the model. Having constructed a sample path, the analyst has already made significant progress into solving the problem of obtaining the *pdf*. Upon observing the sample path diagram, the desired integral equation for the *pdf* of the state variable can be written down by inspection. This follows from a most important property of the level crossing method which often leads to intuitive insights into the model. Namely, every term in the derived integral equation will have a precise mathematical interpretation as a sample-path, state-space level crossing rate, or as a state-space set entrance/exit rate. Combining this term-wise property of the integral equation with conservation laws for long-run up and downcrossing rates of state-space levels, or long-run entrance/exit rates of state-space sets, enables the analyst to write down the desired integral equation for the *pdf* by inspection.

The level crossing method may be viewed as a generalization of the well known *rate in = rate out* principle. This principle is widely used to obtain the steady state distribution of the state variable in continuous time Markov chains having discrete state spaces. The level crossing method, or more generally the system point method, allows us to apply this principle to continuous time stochastic processes with continuous state spaces.

3. Sample Paths

A sample path of the process $\{W(t)\}$ is a single realization of the process over time. Its value at time-point t is an outcome of the random variable $W(t)$, say $X(t)$. We denote an arbitrary sample path by the function $X(t), t \geq 0$, which is real-valued and right continuous on the reals. The function X has jump or removable discontinuities on a sequence of strictly increasing time points (epochs) $\{\tau_n, n = 0, 1, ...\}$, where $\tau_0 = 0$ without loss of generality. Ordinarily, the time points $\{\tau_n\}$ may represent input or output epochs of the content in dams, arrival epochs of customers in queues, or demand or replenishment epochs of stock-on-hand in inventories, etc. Assume that a sample path decreases continuously on time segments between jump points, described by $dX(t)/dt = -r(X(t))$, $\tau_n \leq t < \tau_{n+1}, n = 0, 1, ...$ wherever the derivative exists, and where $r(x) \geq 0$ for all $x \in (-\infty, \infty)$. Note for example, that for the standard virtual wait process in queues, the state space is $[0, \infty)$, $r(x) = 1$ $(x > 0)$ and $r(0) = 0$. In an $< s, S >$ continuous review inventory system with no lead time or backlogging, and the stock on hand decays continuously at constant rate $k \geq 0$, $r(x) = k$ for all $x \in (s, S]$. Here $s \geq 0$ is the reorder point and S is the order-up-to-level. If there is a lead time and backlogging is allowed, the state space is $(-\infty, S]$ and $r(x) = 0$ for $x < s$.

4. **Level Crossings by Sample Paths**

In this article, it is sufficient to consider two types of level crossings from an intuitive viewpoint: *continuous* and *jump* level crossings. A *continuous downcrossing* of level x occurs at a time point $t_0 > 0$ if $\lim_{t \to t_0 -} X(t) = x$ and $X(t) > x$ and is monotone decreasing for all t in a small time interval ending at t_0. Intuitively, one may visualize the sample path as decreasing continuously to level x from above and just reaching level x at the instant t_0. A *jump downcrossing* of level x occurs at a time point $t_0 > 0$ if $\lim_{t \to t_0 -} X(t) > x$ and $X(t_0) \leq x$. Intuitively, one may visualize the sample path as moving strictly above level x for all t in a small time interval ending at t_0, and then jumping vertically downward to a level below x, or to x itself, at the instant t_0.

5. Level Crossings and the Stationary Distribution

This section states without proof, two basic level crossing theorems which greatly assist in writing down an integral equation for the steady state *pdf* g. The results will be stated separately for sample-path downcrossings and sample-path upcrossings. The next section will combine these results with a conservation law for level crossings, to construct the desired integral equation for g.

5.1 Downcrossings

Let $D_t^c(x)$ denote the total number of *continuous* downcrossings of level x and $D_t^j(x)$, the number of *jump* downcrossings of level x during $(0,t)$ due to the external Poisson rate λ_d. The following result holds.

Theorem 5.1. (Brill, 1974, for $r(x) = 1$)

With probability 1

$$\lim_{t\to\infty} D_t^c(x)/t = r(x)g(x) \quad (all\ x) \tag{1}$$

$$\lim_{t\to\infty} D_t^j(x)/t = \lambda_d \int_{y=x}^{\infty} \bar{B_d}(y-x)g(y)dy \quad (allx). \tag{2}$$

Remark 1 *Intuitively, both sides of (1) represent the long-run rate of continuous decays by a typical sample path into level x from above. Both sides of (2) represent the long-run rate of downward jumps which occur at Poisson rate* λ_d*, from state-space set* (x,∞) *into* $(-\infty, x]$.

Remark 2 *Both (1) and (2) hold upon replacing* $D_t^c(x)$ *and* $D_t^j(x)$ *by their expected values, and deleting "with probability 1."*

5.2 Upcrossings

Let $U_t^j(x)$ denote the total number of upcrossings of level x during $(0,t)$ due to the external Poisson rate λ_u. In the present model, these will be jump upcrossings.

Theorem 5.2. (Brill, 1974)

With probability 1,

$$\lim_{t\to\infty} U_t^j(x)/t = \lambda_u \int_{y=-\infty}^{x} \bar{B_u}(x-y)g(y)dy \quad (all\ x). \tag{3}$$

Remark 3 *Intuitively, both sides of (3) represent the long-run rate of upward jumps by a sample path which occur at Poisson rate* λ_u*, from state-space set* $(-\infty, x]$ *into* (x, ∞).

The foregoing theorems relating the rate of continuous level downcrossings of x to the *pdf* g, and relating the rates of jump downcrossings and jump upcrossings of x to integral transforms of the *pdf* g, are an important part of the foundation of the level crossing method. This basis connects with a conservation law of level crossings to construct integral equations, and provides the method with a very strong intuitive appeal.

6. A Conservation Law and an Integral Equation

For every state-space level x and every sample path, the following conservation law holds. In the *long run,*

$$Total\ downcrossing\ rate = Total\ upcrossing\ rate. \tag{4}$$

The foregoing conservation law applies to typical sample paths and every state-space level x. It enables the analyst to write down an integral equation for the *pdf* g in which every term has a precise mathematical interpretation as a long-run rate of sample-path crossings of levels. Thus direct substitution into the above conservation law gives

$$\lim_{t\to\infty} D_t^c(x)/t + \lim_{t\to\infty} D_t^j(x)/t = \lim_{t\to\infty} U_t^j(x)/t. \tag{5}$$

Then, substituting from the above theorems immediately enables us to write down the following integral equation for the *pdf* g. For all x

$$r(x)g(x) + \lambda_d \int_{y=x}^{\infty} \bar{B_d}(y-x)g(y)dy \quad = \lambda_u \int_{y=-\infty}^{x} \bar{B_u}(x-y)g(y)dy. \tag{6}$$

In practice, the procedure starting from a typical sample path and ending with the integral equation for g, is usually carried out quickly and efficiently.

7. Example

Consider a continuous review $< s, S >$ inventory system where $s \geq 0$ is the reorder point and S is the order-up-to level. Assume that demands for stock occur at a Poisson rate λ and demand sizes are *iid* (independent and identically distributed) exponential random variables with mean $1/\mu$. Assume that the stock decays at constant rate $k > 0$ when the stock is in the state-space interval $(s, S]$ and there is no lead time. The ordering policy is: If the stock either decays continuously to, or jumps downward below or to level s, then an order is placed and received immediately, replenishing the stock up to level S.

It is required to derive the steady state *pdf* g of the stock on hand.

Solution: We may specialize the results for the general model given in sections 5. and 6., to this inventory model. Now, the state space is essentially reduced to $(s, S]$, $r(x) = k$, $\lambda_d = \lambda$, and $\lambda_u = 0$. Although $\lambda_u = 0$, the ordering policy ensures that upward jumps – all of them up to level S – occur whenever the stock falls to level s or below s. The rate at which it *decays* to level s is $kg(s)$. The rate at which it *jumps* below level s due to demands is $\lambda \int_{y=s}^{S} e^{-\mu(y-s)} g(y)dy$.

Consider a fixed level $x \in (s, S]$. From (1) and (2), the total *downcrossing* rate of level x is given by

$$kg(x) + \lambda \int_{y=x}^{S} e^{-\mu(y-x)} g(y)dy \quad (s < x \leq S). \tag{7}$$

From the immediately preceding discussion, the total *upcrossing* rate of *every* level $x \in (s, S]$, is precisely equal to the total *downcrosssing* rate of the reorder point, level s. Applying the conservation law for level crossings yields the desired integral equation for g, namely for all $x \in (s, S]$

$$kg(x) + \lambda \int_{y=x}^{S} e^{-\mu(y-x)} g(y)dy \quad (s < x \leq S) = kg(s) + \lambda \int_{y=s}^{S} e^{-\mu(y-s)} g(y)dy. \tag{8}$$

Since all the probability for the stock-on-hand is concentrated on $(s.S]$, the normalizing condition is

$$\int_{x=s}^{S} g(x)dx = 1. \tag{9}$$

Some algebra gives the solution of (8) and (9) simultaneously, for g as

$$g(x) = A \left(1 + \left(\frac{\lambda}{\mu k} \right) e^{\left(-\left(\frac{\lambda}{k} + \mu \right)(S-x) \right)} \right), (s < x \leq S) \tag{10}$$

and $g(x) = 0$ for $x \notin (s, S]$. The constant A in (10) is given by

$$1/A = (S - s) + \frac{\lambda/(k\mu)}{\mu + \lambda/k}\left(1 - e^{\left(-\left(\frac{\lambda}{k}+\mu\right)(S-s)\right)}\right). \tag{11}$$

Notice that g is convex and increasing on $(s, S]$. Intuitively, this implies that most of the time, the stock will reside at relatively high levels, i.e., closer to the order-up-to level S than to the re-order point s. This observation appears to be due to the re-order policy which always replenishes the stock up to level S.

It is interesting to note that if the decay rate $k = 0$, so that the stock remains at a fixed level until the next demand epoch, then there is an atom at level S. Let $\Pi_S = P(\text{inventory is at level } S)$ in the steady state. Then $\forall x \in (s, S)$

$$\begin{aligned} &\lambda\Pi_S \ e^{-\mu(S-x)} + \lambda \int_{y=x}^{S} e^{-\mu(y-x)} g(y)dy \\ &= \lambda\Pi_S \ e^{-\mu(S-s)} + \lambda \int_{y=s}^{S} e^{-\mu(y-s)} g(y)dy \\ &= \lambda\Pi_S \quad (s < x \le S). \end{aligned} \tag{12}$$

with normalizing condition $\Pi_S + \int_s^S g(x)dx = 1$. It is then readily shown that g is uniform on (s, S), and there is an atom at S, given by

$$\Pi_S = \frac{1}{1 + \mu(S - s)}, \quad g(x) = \frac{\mu}{1 + \mu(S - s)}, \ x \in (s, S). \tag{13}$$

It is also interesting to note that the total *ordering rate* is given by the total *downcrossing* rate of level s, which is just the right hand side of (8) when $k > 0$, or the right hand side of (12) when $k = 0$.

8. Summary

This article presents an overview of the level crossing method for a fairly general storage model. For expository purposes, an example is also presented which applies the method to a particular, basic, very well known inventory system. It is emphasized that the level crossing method equally applies to a vast array of other stochastic models as well. It would have been equally instructive to have presented an example highlighting any one of them. The level crossing method applies to other storage models with limited capacity, blocked input rules, a variety of state dependent control policies, etc. It applies to both simple and extremely complex queueing systems, such as $M/G/1$, $M/M/c$, $G/M/1$, $G/M/c$ queues with reneging, bounded virtual wait or workload, server vacations, priorities, and a host of possible state dependencies. In fact, the method was discovered in the context of queues, as mentioned in the introduction.

9. References

1. Azoury, K. and Brill, P.H. (1986), "An Application of the System-Point Method to Inventory Models under Continuous Review", *J. Applied Probability* **23**, 778-789.

2. Brill, P.H. (1975), *System Point Theory in Exponential Queues*, Ph.D. Dissertation, University of Toronto.

3. Brill, P.H. (1979), "An Embedded Level Crossing Technique for Dams and Queues", *Applied Probability* **16**, 174-186.

4. Brill, P.H. (1996), "Level Crossing Methods", in *Encyclopedia of Operations Research and Management Science*, Gass, S.I. and Harris, C.M.. editors, Kluwer Academic Publishers, 338-340.

5. Brill, P.H. and Posner, M.J.M. (1977), "Level Crossings in Point Processes Applied to Queues: Single Server Case", *Operations Research* **25**, 662-673.

6. Brill, P.H. and Posner M.J.M. (1981), "The System Point Method in Exponential Queues: A Level Crossing Approach", *Mathematics of Operations Research* **6**, 31-49.

7. Brill, P.H. and Harris, C.M. (1997), "M/G/1 Queues with Markov-generated Server Vacations", *Stochastic Models*, **13(3)**, 491-521.

8. Harris, C.M., Brill, P.H., and Fischer M. (2000), "Internet-Type Queues with Power-Tailed Interarrival Times and Computational Methods for their Analysis", *Informs J. on Computing*, to appear.

9. Lindley, D.V. (1952), "The Theory of Queues with a Single Server", *Proc. Cambridge Philosophical Soc.*, **48,** 277-289.

10. Miyazawa, M. (1994), "Rate Conservation Laws: A Survey", *QUESTA*, **18**, 1-58.

11. Ross, S. (1996), *Introduction to Probability Models*, *6*th edition. Academic Press, Inc.